1977-05

Coupled Azimuthal Potentials for Electromagnetic Field Problems in Inhomogeneous Axially Symmetric Media

Morgan, Michael A.

[6] M. I. Kontorovich, V. Yu. Petrun'kin, N. A. Yesepkina, and M. I. Astrakhan, "The coefficient of reflection of a plane electromagneitc wave from a plane wire mesh," *Radio Engr. and Elect. Phys.*, vol. 7, pp. 221–331, Feb. 1962.
[7] N. M. Zolotukhina, "Averaged boundary conditions for a two-dimensional slot array," *Radio Engr. Elect. Phys.*, vol. 20, pp. 112–114, Mar. 1975.
[8] G. J. van den Broek and J. van der Vooren, "Reflection properties of periodically supported metallic wire gratings with rectangular mesh showing small sag," *IEEE Trans. Antennas Propagat.*, vol. AP-19, pp. 109–113, Jan. 1971.

Coupled Azimuthal Potentials for Electromagnetic Field Problems in Inhomogeneous Axially Symmetric Media

MICHAEL A. MORGAN, STUDENT MEMBER, IEEE,
SHU-KONG CHANG, STUDENT MEMBER, IEEE,
AND KENNETH K. MEI, MEMBER, IEEE

***Abstract*—Classical electromagnetic potential formulations are, with the exceptions of a few special cases of one-dimensional stratification, restricted to use in uniform media. A recently developed potential formulation that provides a flexible basis for numerical computation of time-harmonic field problems involving continuously and discretely inhomogeneous axially symmetric media is the topic of this paper. The formulation manifests itself in both a differential equation system and, alternately, a variational criterion. Typical numerical applications include solutions of scattering by arbitrarily shaped material bodies of revolution and radiation from inhomogeneously loaded rotationally symmetric antenna structures. Current numerical investigations by the authors, using Mei's unimoment method in conjunction with both finite-difference and finite-element techniques, have shown the formulation to be highly feasible for computation of field problems having dimensions as large as several wavelengths.**

I. INTRODUCTION

Numerical computation of time-harmonic electromagnetic boundary value problems has recently been extended into the regime of inhomogeneous axially symmetric media by way of the potential formulation to be presented in this paper. This formulation provides the basis for the numerical solution of an important class of interior, radiation, and scattering problems, involving inhomogeneous bodies of revolution having multiwavelength dimensions, which were heretofore unapproachable using conventional techniques.

Electromagnetic fields are traditionally represented in terms of potentials for ease of analysis. Potential formulations also offer numerical advantages by usually having fewer coupled unknowns and a higher order of continuity than the original electromagnetic field. Computation time is, in fact, proportional to the cube of the number of coupled scalar fields which are present as unknowns in the formulation. A general vector potential representation may be developed for completely arbitrary inhomogeneous isotropic material, but the formulation is in terms of four coupled scalar fields [1]. Without potentials, Maxwell's equations must be solved using at least three coupled scalar functions: namely the vector of either the electric or the magnetic field, which are in general discontinuous at material interfaces. Classical TE and TM potential formulations generate all fields from only two uncoupled scalar functions, but are only valid for uniform media and some special cases of separable coordinate one-dimensional inhomogeneities, such as that of a spherically stratified media [1], [2].

Manuscript received February 23, 1976. This work was sponsored by U.S. Army Mobility Equipment Research and Development Center Contract DAAK02-75-C-002.

The authors are with the Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, CA 94720.

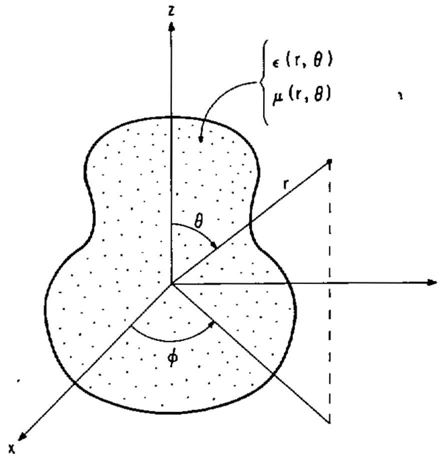

Fig. 1. Axially symmetric inhomogeneous medium.

The potential formulation to be presented here is valid in generally inhomogeneous isotropic rotationally symmetric media, as shown in Fig. 1, where the constitutive parameters $\epsilon(\bar{r})$ and $\mu(\bar{r})$ are invariant to the azimuthal coordinate ϕ. All electromagnetic field components are expanded via exponential Fourier series in the ϕ-coordinate. The modal electromagnetic fields are shown to be representable by two coupled azimuthal potentials (CAP's), so named because of their one-to-one relationship with the azimuthal components of the modal electric and magnetic fields. The potentials in the CAP formulation, which are everywhere continuous, are then shown to satisfy a system of coupled partial differential equations and, alternately, a variational criterion, either of which can be used as a basis for developing numerical solution algorithms.

The range of possible application of this new potential representation of Maxwell's equations includes the numerical solutions of both interior region Dirichlet type problems and open region radiation and scattering problems, using Mei's "unimoment method" [3]. The CAP formulation can be readily employed, in conjunction with appropriate numerical techniques, to solve such typical problems as scattering by distorted raindrops, power absorption by azimuthally invariant biological tissue models, and effects of inhomogeneous dielectric loading on axially symmetric antenna structures and waveguide sections.

II. MODAL FIELD DECOMPOSITION AND GENERATING EQUATIONS

The normalized coordinates to be used in the subsequent equations are defined by $(R, Z, \phi) = (k_0\rho, k_0z, \phi)$, where (ρ, z, ϕ) are standard circular cylindrical coordinates and $k_0 = 2\pi/\lambda_0$ is the free-space wavenumber of the time-harmonic field. The reason for this particular choice of coordinates is twofold. First, the use of this normalized coordinate system yields very compact equation presentations, and second, these coordinates form a two-dimensional Cartesian system in the spatial solution domain for the CAP formulation. This latter property is of particular value for use with the finite-element method, where numerical surface integrations must be performed over the two-dimensional spatial solution domain. The relative constitutive parameters of the medium, $\epsilon_r(R, Z)$ and $\mu_r(R, Z)$, are invariant to the ϕ-coordinate.

The equations of the CAP formulation are obtained by first decomposing the total fields into azimuthal modes via an exponential Fourier series in the ϕ-coordinate

$$\bar{E}(R, Z, \phi) = \sum_{m=-\infty}^{\infty} \bar{e}_m(R, Z) \exp(jm\phi) \tag{1a}$$

$$\eta_0 \bar{H}(R, Z, \phi) = \sum_{m=-\infty}^{\infty} \bar{h}_m(R, Z) \exp(jm\phi) \tag{1b}$$

where $\eta_0 = 120\pi\ \Omega$. Due to the assumed axial symmetry of the medium, the modal fields have time-average power orthogonality and decouple when the field expansions in (1) are substituted into Maxwell's time-harmonic source-free equations, yielding the first-order coupled system given by

$$\epsilon_r e_{\phi,m} = j\left[\frac{\partial h_{Z,m}}{\partial R} - \frac{\partial h_{R,m}}{\partial Z}\right] \tag{2a}$$

$$\mu_r h_{\phi,m} = j\left[\frac{\partial e_{R,m}}{\partial Z} - \frac{\partial e_{Z,m}}{\partial R}\right] \tag{2b}$$

$$R\epsilon_r e_{R,m} = j\left[\frac{\partial(Rh_{\phi,m})}{\partial Z} - jmh_{Z,m}\right] \tag{2c}$$

$$R\mu_r h_{R,m} = j\left[jme_{Z,m} - \frac{\partial(Re_{\phi,m})}{\partial\ \partial Z}\right] \tag{2d}$$

$$R\epsilon_r e_{Z,m} = j\left[jmh_{R,m} - \frac{\partial(Rh_{\phi,m})}{\partial R}\right] \tag{2e}$$

$$R\mu_r h_{Z,m} = j\left[\frac{\partial(Re_{\phi,m})}{\partial R} - jme_{R,m}\right] \tag{2f}$$

where the azimuthal dependence of the modal fields is suppressed. Using judicious substitution between (2c) and (2f), and between (2d) and (2e), it is easily found that all modal field components can be generated from two modal scalar potential functions $\psi_1(R, Z, m)$ and $\psi_2(R, Z, m)$, via

$$\hat{\phi} \times \bar{e}_m = jf_m(m\hat{\phi} \times \nabla\psi_1 - R\mu_r\nabla\psi_2) \tag{3a}$$

$$\hat{\phi} \cdot \bar{e}_m = \psi_1/R \tag{3b}$$

$$\hat{\phi} \times \bar{h}_m = jf_m(m\hat{\phi} \times \nabla\psi_2 + R\epsilon_r\nabla\psi_1) \tag{3c}$$

$$\hat{\phi} \cdot \bar{h}_m = \psi_2/R \tag{3d}$$

where $\hat{\phi}$ is the unit azimuthal vector, the two-dimensional gradient operator is defined by

$$\nabla = \hat{R}\frac{\partial}{\partial R} + \hat{Z}\frac{\partial}{\partial Z} \tag{4a}$$

and

$$f_m(R, Z) = [\mu_r(R, Z)\epsilon_r(R, Z)R^2 - m^2]^{-1}. \tag{4b}$$

As was previously mentioned, the normalized cylindrical coordinates (R, Z, ϕ) were chosen only for the practical reasons of compact presentation and utility in finite-element numerical computations. The equations of the CAP formulation can be easily obtained in an arbitrary coordinate system (u_1, u_2, ϕ), where u_1 and u_2 are both orthogonal to ϕ and form some single-valued system of coordinates in any constant azimuth planar cross section of the region of revolution. For example, the equations throughout this paper can be trivially recast into standard spherical coordinates (r, θ, ϕ) using the simple substitutions $R = k_0 r \sin\theta$, $Z = k_0 r \cos\theta$, and $\nabla = [(1/k_0)\hat{r}(\partial/\partial r) + \hat{\theta}(1/r)(\partial/\partial\theta)]$.

A very important property of the potentials can be obtained easily by noting that since the modal potentials are proportional to the ϕ-components of $\bar{e}_m$ and $\bar{h}_m$, as shown in (3b) and (3d), they are continuous everywhere, including at dielectric and magnetic interfaces. This property of uniform field continuity is very desirable in numerical computations. For the case of the variational finite-element method the utility of the potential field continuity is particularly pronounced, in that material interfaces require no supplemental boundary conditions and are handled naturally with the same general algorithm as is continuously inhomogeneous media.

III. DIFFERENTIAL EQUATION REPRESENTATION

The partial differential equations obeyed by the coupled azimuthal potentials ψ_1 and ψ_2 may be obtained directly from Maxwell's equations by substitution of the modal field generating equations in (3) into (2a) and (2b). The potentials are found to satisfy a system of coupled second-order linear partial differential equations given by

$$\nabla \cdot [f_m(R\epsilon_r\nabla\psi_1 + m\hat{\phi} \times \nabla\psi_2)] + \epsilon_r\psi_1/R = 0 \tag{5a}$$

$$\nabla \cdot [f_m(R\mu_r\nabla\psi_2 - m\hat{\phi} \times \nabla\psi_1)] + \mu_r\psi_2/R = 0 \tag{5b}$$

where the gradient operator ∇ is as defined previously in (4a) and $f_m(R, Z)$ is given by (4b).

To utilize the unimoment technique in obtaining solutions to open region radiation and scattering problems requires the ability to solve interior Dirichlet type boundary value problems within a closed region containing the inhomogeneities of the problem, (e.g., an antenna structure or scattering bodies). In solving such an interior Dirichlet problem the two-dimensional solution domain of (5) is a planar constant-ϕ cross section of an interior volume of revolution, as is illustrated in Fig. 2. The interior region S, which is bounded by the curve C, contains inhomogeneous material specified by the relative constitutive parameters $\epsilon_r(R, Z)$ and $\mu_r(R, Z)$. The surface S could, for example, be the semicircular cross section of a geometrical sphere which contains an arbitrarily shaped, (and possibly inhomogeneous), material body of revolution, such as a torus.

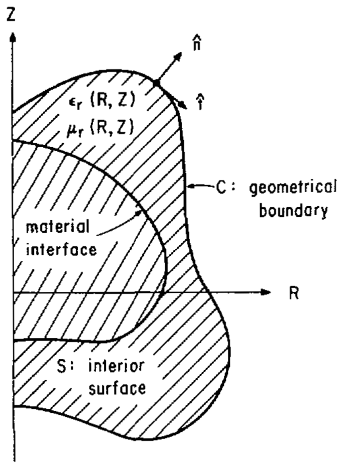

Fig. 2. Constant aximuth planar cross section.

The question of uniqueness of the solution to the coupled system in (5), for arbitrary Dirichlet boundary conditions of the potentials specified on C, may be resolved by using the standard proof for electromagnetic field uniqueness assuming nonzero losses [4], where it is noted that ψ_1 and ψ_2 are proportional to the ϕ-components of $\overline{e}_m$ and $\overline{h}_m$, respectively.

For the special case of azimuthally invariant modal fields ($m = 0$), the differential equations in (5) decouple, giving a simple potential formulation that was originally derived by Abraham and is given more currently by Stratton [5]. This uncoupled formulation was the basis for a recently completed investigation of inhomogeneous dielectric loading of biconical antenna structures, conducted by Stovall and Mei, using finite-difference simulation of the PDE for ψ_2 when $m = 0$ [6]. It should be noted that only for the case of the axially symmetric mode ($m = 0$), can the total modal field be decomposed into partial fields which are either TE or TM to $\hat{\phi}$. This constraint expresses the fact that no azimuthally changing modal electromagnetic field ($m \neq 0$) can be generated having a null azimuthal component of either $\overline{e}_m$ or $\overline{h}_m$ throughout an axially symmetric source-free volume of revolution. This result is valid even when ϵ and μ are constant, which is perhaps the reason why the CAP formulation has been overlooked by classical electromagnetic analysts.

To further investigate the nature of the system of CAP PDE's in (5) we will use matrix operator notation to recast the equations into a more compact form. Defining the solution vector

$$\psi(R, Z, m) = \begin{bmatrix} \psi_1(R, Z, m) \\ \psi_2(R, Z, m) \end{bmatrix} \tag{6}$$

and the matrix differential operator

$$A(R, Z, m) = \begin{bmatrix} \nabla \cdot (f_m R \epsilon_r \nabla) + \epsilon_r/R & m\nabla \cdot (f_m \hat{\phi} \times \nabla) \\ -m\nabla \cdot (f_m \hat{\phi} \times \nabla) & \nabla \cdot (f_m R \mu_r \nabla) + \mu_r/R \end{bmatrix} \tag{7}$$

the system in (5) is given by

$$A\psi = \mathbf{0} \tag{8a}$$

with Dirichlet BC's on C given by

$$\psi(R, Z, m)\,|_C = \beta(c). \tag{8b}$$

A very important property of the operator A is its formal self-adjointness on the space of vector functions having homogeneous Dirichlet BC's on C,

$$\langle u, Av \rangle = \langle v, Au \rangle \tag{9}$$

where the nonconjugated inner dot product is defined by

$$\langle a, b \rangle = \int_S a(R, Z) \cdot b(R, Z)\, dR\, dZ \tag{10}$$

and the vector functions in (9) satisfy the adjoint boundary condition $u\,|_C = v\,|_C = \mathbf{0}$. As a consequence of this operator self-adjointness it follows directly that a variational principle exists for the CAP formulation.

IV. VARIATIONAL FORMULATION

The numerical solution of the system of PDE's in (5) via finite-difference simulation requires the use of basis functions which are twice differentiable. This continuity constraint on the basis functions may be relaxed to only first-order differentiability if a variational formulation is used in conjunction with the finite-element method.

In establishing a variational principle we will follow two separate, but equivalent, routes. Our first avenue is the Euler–Lagrange formulation which provides the simplest, but somewhat indirect, approach to a variational criterion. The second development relies upon the self-adjointness of the system operator A in conjunction with the application of a stationary theorem, to provide direct access to a variational principle.

The Euler–Lagrange variational formulation is based upon the existence of a functional of the potentials and their first derivatives which is stationary about the correct solution of (5) for given Dirichlet BC's on the boundary curve C of Fig. 2. The functional manifests itself as a surface integral over the planar cross section surface S,

$$F = \int_S L(R, Z, \psi_1, \psi_2, \nabla\psi_1, \nabla\psi_2)\, dR\, dZ \tag{11}$$

It is well known from the calculus of variations that at the stationary point of the functional F the Lagrangian L satisfies a system of coupled Euler–Lagrange partial differential equations [7],

$$\nabla \cdot \left[\frac{\partial L}{\partial(D_R \psi_k)} \hat{R} + \frac{\partial L}{\partial(D_Z \psi_k)} \hat{Z} \right] - \frac{\partial L}{\partial \psi_k} = 0, \quad k = 1, 2. \tag{12}$$

The foremost problem in deriving such a variational criterion, using an original differential equation formulation, is to discover the Lagrangian L which when substituted into (12) will yield the original system of PDE's. This unearthing of the Lagrangian is not always a simple matter, particularly in cases of greater than second-order PDE's where higher order derivative terms appear in the Lagrangian and the Euler–Lagrange system is more complicated than in (12).

The discovery of the Lagrangian for the CAP formulation is most quickly performed by a detailed inspection process, which yields,

$$L = f_m [\nabla\psi_1 \cdot (R\epsilon_r \nabla\psi_1 + m\hat{\phi} \times \nabla\psi_2) + \nabla\psi_2 \cdot (R\mu_r \nabla\psi_2 - m\hat{\phi} \times \nabla\psi_1)] - (\epsilon_r \psi_1^2 + \mu_r \psi_2^2)/R. \tag{13}$$

This result, which is unique to within an arbitrary constant multiplier and independent additive function, may be verified by direct substitution into (12).

The second approach to establishing a variational principle is via a "stationary theorem" applicable to formally self-adjoint operators [8]. This stationary theorem is directly extendable to the matrix operator equation of (8). In particular, because of the self-adjoint nature of A, if we have a boundary value problem of the type,

$$\mathsf{A}\Psi = \Phi \tag{14a}$$

with homogeneous Dirichlet BC's

$$\Psi(R, Z, m)\,|_C = 0 \tag{14b}$$

where $\Phi(R, Z, m)$ is some known driving vector, then a functional that is stationary about the correct solution of (14) is given by

$$G = 2\langle \Psi, \Phi\rangle - \langle \Psi, \mathsf{A}\Psi\rangle \tag{15}$$

where the inner products are defined in (10).

The conversion of the CAP system in (8) to one of the form in (14) is accomplished by the simple change of variables,

$$\Psi(R, Z, m) = \psi(R, Z, m) - \beta(R, Z) \tag{16a}$$

$$\Phi(R, Z, m) = -\mathsf{A}\beta(R, Z) \tag{16b}$$

where $\beta(R, Z)$ is any arbitrary vector function satisfying the same boundary conditions on C as does $\psi(R, Z, m)$. Substituting (16) into (15) yields

$$G = -\langle \psi, \mathsf{A}\psi\rangle - \langle \psi, \mathsf{A}\beta\rangle + \langle \beta, \mathsf{A}\psi\rangle + \langle \beta, \mathsf{A}\beta\rangle. \tag{17}$$

Using direct vector calculus operations on the combination of the second and third inner products, which is commonly referred to as the bilinear concomitant of ψ and β, it will be found that G can be written as

$$\begin{aligned} G = \{&-\langle \psi, \mathsf{A}\psi\rangle + \oint_C \psi_1 f_m (R\epsilon_r \nabla\psi_1 + m\hat{\phi} \times \nabla\psi_2) \\ &+ \psi_2 f_m (R\mu_r \nabla\psi_2 - m\hat{\phi} \times \nabla\psi_1) \cdot \hat{n}\,|dc|\} \\ &- \Big\{-\langle \beta, \mathsf{A}\beta\rangle + \oint_C \beta_1 f_m (R\epsilon_r \nabla\beta_1 + m\hat{\phi} \times \nabla\beta_2) \\ &+ \beta_2 f_m (R\mu_r \nabla\beta_2 - m\hat{\phi} \times \nabla\beta_1) \cdot \hat{n}\,|dc|\Big\}. \end{aligned} \tag{18}$$

This can be recast into a more compact form by noting that the terms within each set of curly brackets can be recombined to yield a surface integral of the exact form of the Euler-Lagrange functional in (11) and (13). The result is

$$\begin{aligned} G = &\int_S L(R, Z, \psi_1, \psi_2, \nabla\psi_1, \nabla\psi_2)\, dR\, dZ \\ &- \int_S L(R, Z, \beta_1, \beta_2, \nabla\beta_1, \nabla\beta_2)\, dR\, dZ = F_\psi - F_\beta. \end{aligned} \tag{19}$$

The stationary functionals G and F, as given in (17) and (11), respectively, thus differ only by a functional of the arbitrary and independent vector $\boldsymbol{\beta}$ and its derivatives: namely F_β, whose first variation with respect to the solution vector ψ is zero. The Euler-Lagrange and stationary theorem approaches thus yield stationary functionals which are precisely equivalent in the variational sense.

In numerical applications of this variational criterion, using the finite-element method, it is often convenient to change variables and select as the fundamental unknowns the ϕ-components of the modal fields, $e_{\phi,m}(R, Z) = \psi_1/R$ and $h_{\phi,m}(R, Z) = \psi_2/R$. If linear basis functions are used to approximate the potentials ψ_1 and ψ_2 then the last quadratic term in the Lagrangian in (8), $(\epsilon_r\psi_1{}^2 + \mu_r\psi_2{}^2)/R$, will introduce a nonintegrable infinite singularity along the Z axis, where $R = 0$. The change of variables previously mentioned eliminates this singularity effect, but it then becomes necessary to know the boundary conditions obeyed by $e_{\phi,m}$ and $h_{\phi,m}$ along the section(s) of the Z axis which bound the surface S. These boundary conditions can be obtained via a judicious cross substitution and limiting procedure using Maxwell's equations, and are given here for reference purposes:

$$e_{\phi,m}(R, Z)\,|_{R=0} = h_{\phi,m}(R, Z)\,|_{R=0} = 0, \quad \text{for } m^2 \neq 1 \tag{20a}$$

$$3\frac{\partial h_{\phi,m}}{\partial R} + \left(\frac{\partial \ln \epsilon_r}{\partial R}\right) e_{\phi,m}\Bigg|_{R=0} = 0, \quad \text{for } m^2 = 1 \tag{20b}$$

$$3\frac{\partial h_{\phi,m}}{\partial R} + \left(\frac{\partial \ln \mu_r}{\partial R}\right) h_{\phi,m}\Bigg|_{R=0} = 0, \quad \text{for } m^2 = 1. \tag{20c}$$

The homogeneous Dirichlet BC's for $m^2 \neq 1$, in (20a), appear as an obvious result of the continuity and single-valuedness of the vector EM field as the Z axis is approached from all constant azimuth paths. The Cauchy BC's for the case of $m^2 = 1$, in (20b) and (20c), reduce to homogeneous Neumann BC's, (radial derivatives equal to zero), for the case of media composed of uniform material bodies of revolution.

A final important topic associated with the variational formulation concerns the physical interpretation of the stationary functional F, as displayed in (11) and (13). Using simple substitutions from (3), it can be shown that the Lagrangian in (13) can be written in terms of the modal vector electromagnetic field as

$$L = -R(\epsilon_r\bar{e}_m \cdot \bar{e}_m + \mu_r\bar{h}_m \cdot \bar{h}_m) - 2m\hat{\phi} \cdot (\bar{e}_m \times \bar{h}_m). \tag{21}$$

A direct vector manipulation of Maxwell's curl equations for the modal fields will show that this can be alternately written as

$$L = -jR\nabla' \cdot [(I - \hat{\phi}\hat{\phi}) \cdot (\bar{e}_m \times \bar{h}_m)] \tag{22}$$

where I is the identity dyadic and $\nabla' \cdot$ is the conventional three-dimensional divergence operating on the transverse (to $\hat{\phi}$) vector components of the "pseudo" Poynting vector, $\bar{e}_m \times \bar{h}_m$. Upon substituting (22) into (13), the divergence theorem can be used to transform the surface integral of F into a line integral around C of the outward normal of the "pseudo" Poynting vector:

$$F = -j\oint_C (\bar{e}_m \times \bar{h}_m) \cdot \hat{n}R\,|dc|. \tag{23}$$

The term "pseudo" is used because no conjugates appear in the cross products in the integrand, as would be the case if the terms were related to time-average power. The physical significance of the stationary functional F can be displayed by considering the time-varying power per unit azimuth radian of the mth order modal field that is radiated outward from a constant-ϕ planar cross section. This time-varying power density can be written as

$$\frac{\partial P(\phi, t)}{\partial \phi} = \frac{1}{2\eta_0 k_0{}^3} \operatorname{Re}\left\{ \oint_C (\bar{e}_m \times \bar{h}_m{}^*) \cdot \hat{n}R \mid dc \mid + \left[\oint_C (\bar{e}_m \times \bar{h}_m) \cdot \hat{n}R \mid dc \mid \right] e^{j2(\omega t + m\phi)} \right\} \quad (24)$$

where the integral containing the magnetic field conjugate $\bar{h}_m{}^*$ represents time-average power while the term in square brackets is the stationary functional F. The functional F is clearly associated with the difference of the time-varying and time-average radiated power densities. The exact physical interpretation of the stationarity of F appears somewhat illusive and the quest for such an interpretation certainly warrants further consideration.

V. CONCLUSION

The CAP formulation provides an important foundation for performing efficient numerical computations of solutions to the often encountered class of field problems involving axially symmetric media. Using the unimoment method, scattering and radiation problems can be handled very effectively by coupling exterior radiation field series solutions to interior region Dirichlet boundary value problems, which are solved either through finite-difference simulation of the CAP differential equation system [8], or through use of the variational finite-element technique.

In addition to the CAP formulation there have been discovered two alternate coupled scalar potential representations that utilize as potentials either the two vector components of $\hat{\phi} \times \bar{e}_m$ or the components of $\hat{\phi} \times \bar{h}_m$. There is, however, apparently no Euler–Lagrange functional for these formulations.

The authors currently have a very successful and operational numerical code which solves for EM scattering by arbitrarily shaped, and generally lossy, dielectric and magnetic bodies of revolution using the variational finite-element method. A detailed description of the implementation of this numerical method, along with extensive experimental and analytical verification, will be the topic of a forthcoming paper.

REFERENCES

[1] J. Van Bladel, *Electromagnetic Fields.* New York: McGraw-Hill, 1964, pp. 229–250.
[2] C. T. Tai, "The electromagnetic theory of the spherical luneberg lens," *Appl. Sci. Res.*, sec. B, vol. 7, pp. 133–130, 1959.
[3] K. K. Mei, "Unimoment method of solving antenna and scattering problems," *IEEE Trans. Antennas Propagat.*, vol. AP-22, pp. 760–766, Nov. 1974.
[4] R. F. Harrington, *Time-Harmonic Electromagnetic Fields.* New York: McGraw-Hill, 1961, pp. 100–103.
[5] J. A. Stratton, *Electromagnetic Theory.* New York: McGraw-Hill, 1941, p. 422.
[6] R. E. Stovall and K. K. Mei, "Application of a unimoment technique to a biconical antenna with inhomogeneous dielectric loading," *IEEE Trans. Antennas Propagat.*, vol. AP-23, pp. 335–341.
[7] P. M. Morse and H. Feshback, *Methods of Theoretical Physics.* New York: McGraw-Hill, 1953, pp. 275–280.
[8] I. Stakgold, *Boundary Value Problems of Mathematical Physics*, Vol. 2. London: Macmillan, 1968, ch. 8.
[9] M. A. Morgan and K. K. Mei, "Numerical computation of E. M. scattering by inhomogeneous bodies of revolution," *Abstracts for the 1974 URSI Symposium on E.M. Wave Theory*, London, England, July 1974.
[10] G. Strang and G. J. Fix, *An Analysis of the Finite Element Method.* Englewood Cliffs, NJ: Prentice-Hall, 1974.
[11] S. K. Chang, M. A. Morgan, and K. K. Mei, "Coupled potential formulation for 3-D E.M. boundary value problems in inhomogeneous axially symmetric media," *Abstracts for the 1975 IEEE/AP-S Symposium*, Urbana, IL, June 1975.

Some Extensions of Babinet's Principle in Electromagnetic Theory

THOMAS B. A. SENIOR, FELLOW, IEEE

Abstract—**The concept of resistive and conductive sheets provides a meaningful extension of Babinet's principle to surfaces which are no longer perfect. The complementary problems are described, and the appropriate field relations derived.**

Babinet arrived at the principle, which now bears his name, by comparing the diffraction pattern of an aperture with that of a complementary disk. It was later verified for scalar waves subject to a Neumann or Dirichlet boundary condition on the screen or disk, and extended to electromagnetic waves by Booker [1] who pointed out the polarization rotation of the primary field which is necessary if the screen and disk are both perfectly conducting.

These known forms of Babinet's principle are all consequences of the symmetry of the fields radiated by planar distributions and (where appropriate) the duality of electromagnetic fields. Over the years there have been a number of attempts to extend the principle to surfaces which are not perfect, for example, by Neugebauer [2] to surfaces which are absorbing, and more recently by Lang [3] to resistive surfaces. In Lang's extension the complementary structures are a perfectly conducting screen with a resistive insert and a resistive screen with a perfectly conducting insert, but the derivation has been criticized [4] for the assumptions made concerning the normal components of the field. Nevertheless, as noted by Baum and Singaraju [5], resistive sheets and their electromagnetic duals do afford an exact extension of Babinet's principle. This fact was exploited by Senior [6] in developing some generalized forms in acoustics, and we here discuss the analogous results for electromagnetic waves.

Manuscript received May 4, 1976; revised August 22, 1976. This work was supported by the U.S. Air Force Office of Scientific Research under Grant 72-2262.

The author is with the Radiation Laboratory, University of Michigan, Ann Arbor, MI 48109.